Quiz 5 Solutions

1. A powdered [redacted]
with monochrom[redacted] at a
measured angle of 25 [redacted].
a. What is the energy of a [redacted]?
The sample is now exposed to a [redacted].
b. What is the kinetic energy of a [redacted] at the
same angle?
c. What is the kinetic energy of a [redacted]?

a) Bragg diffraction: $2d\sin\theta = n\lambda$.

$n=1,\ d=0.25\,nm,\ \theta=25^\circ \Rightarrow \lambda = 2\times 0.25\times \sin 25^\circ = 0.211\,nm$

For a photon, $E = \dfrac{hc}{\lambda} = \dfrac{1240\,eV\,nm}{0.211\,nm} = 5877\ eV$

b) For a neutron, same n, same θ, same d $\Rightarrow \lambda = 0.211\,nm$ also.

de Broglie: $p = \dfrac{h}{\lambda}$ so $pc = \dfrac{hc}{\lambda} = 5877\ eV$ as above

Now $pc \ll m_n c^2$ (938 MeV) so neutrons are non-relativistic

$\therefore$ We can use kinetic energy $K = \dfrac{p^2}{2m_n} = \dfrac{(pc)^2}{2m_n c^2}$

$$K = \frac{(5877\,eV)^2}{2\times 938\times 10^6\,eV} = \underline{0.018\,eV}$$

c) For $n=2$, $2\lambda = 2d\sin\theta \Rightarrow \lambda = 0.105\,nm = \frac{1}{2}$ original λ above $(\lambda_1$

As before, $K = \dfrac{p^2}{2m_n} = \dfrac{(pc)^2}{2m_n c^2} = \dfrac{h^2c^2}{2m_n c^2\lambda^2}$

Notice $K \propto \dfrac{1}{\lambda^2}$ so if $\lambda = \frac{1}{2}\lambda_1$, $K = 4K_1$ with $K_1 = 0.018\,eV$

$\Rightarrow \underline{K = 0.074\,eV}$

2. Protons and electrons were known to exist in nuclei many years before discovery of the neutron. In fact, Rutherford theorized that the helium nucleus (Z=2) is made of 2 electrons and 4 protons bound together by electrostatic forces. We can now use the Heisenberg Uncertainty Principle to see if this is reasonable. Take the helium nuclear radius to be $r_{nuc} = 10^{-14}$m.

a. Estimate the possible value of kinetic energy (in eV) for an electron in such a nucleus. (Hint: the electron is moving at close to light speed, so total energy is related to momentum by $E \approx pc$ - you'll get extra credit for writing down the exact formula!).

b. What is the attractive (therefore negative) Coulomb potential energy (in eV) at the surface of the nucleus, charge +2e? Therefore, is the electrostatic attraction strong enough to prevent such an electron escaping?

a) Heisenberg : $\Delta p \, \Delta x \approx \hbar$. Here $\Delta x \approx r_{nuc} = 10^{-14}\text{m} = 10^{-5}\text{nm}$.

$\therefore \; \Delta p \approx \dfrac{h}{2\pi \Delta x} \rightarrow \Delta pc \approx \dfrac{hc}{2\pi r_{nuc}} = \dfrac{1240\text{eVnm}}{2\pi \times 10^{-5}\text{nm}} = 19.73\text{MeV}$

In general, total energy $E = \sqrt{(m_e c^2)^2 + (pc)^2}$. Here $m_e c^2 = 0.511\text{MeV} \ll pc$

$\therefore \; E \approx pc$. (highly relativistic). Kinetic energy $K = E - m_e c^2$

i.e. $K \approx 19.73\text{ MeV} - 0.511\text{MeV} = \underline{19.2\text{ MeV}}$

b) Potential energy $U = -\dfrac{Ze^2}{4\pi\varepsilon_0 r_{nuc}} = -2 \cdot \dfrac{1.440\text{eVnm}}{10^{-5}\text{nm}} = -288\text{keV}$

$\therefore$ electron's kinetic energy >> "escape energy" $|U|$, so electron <u>cannot be confined</u>.

<u>Note:</u> For any r, as long as electron is relativistic $K \approx E \approx pc$

$\Rightarrow K \approx \dfrac{\hbar c}{r} = \dfrac{197\text{eV.nm}}{r}$. Potential energy $U = \dfrac{-2.1.440\text{eVnm}}{r} = \dfrac{-2.88\text{eV}}{r}$

$\therefore$ electron is not confined at any radius !

However, further out, $K = \dfrac{p^2}{2m} \approx \dfrac{(\hbar c)^2}{2 m_e c^2 r^2} = \dfrac{0.04\text{ eV}}{r^2}$

$\Rightarrow$ K falls off faster than $|U|$. Use $2K + U = 0$ to solve for r

$\Rightarrow$ stable orbit at $r \sim 0.1\text{nm}$.

3. An atom in a gas [redacted] ground state. It [redacted] ground state by [redacted]

a. What is the [redacted]
b. What is the [redacted]
c. Therefore, [redacted] approximate [redacted] use basic [redacted]).

a) $\lambda = \dfrac{hc}{E} = \dfrac{1240\,\text{eV nm}}{1.8\,\text{eV}} = \underline{688.9\text{ nm}}$ (red)

b) Lifetime of state $\Delta t = 0.1\text{ns}$, Heisenberg $\Rightarrow \Delta E \Delta t \approx \hbar$

$$\Rightarrow \Delta E = \frac{6.6\times10^{-34}\text{ Js}}{2\pi \times 0.1\times10^{-9}\text{s}} = 1.05\times10^{-24}\text{ J} = \underline{6.56\text{ meV}}$$

c) Using $\lambda = hc/E$:

Hard way: Evaluate $\lambda_{hi} = \dfrac{hc}{1.8+6.56\times10^{-3}}$, $\lambda_{low} = \dfrac{hc}{1.8-6.56\times10^{-3}}$

then $\Delta\lambda = \lambda_{hi} - \lambda_{low}$

Easy way: $\Delta\lambda = \dfrac{\partial\lambda}{\partial E}\Delta E = -\dfrac{hc}{E^2}\Delta E \Rightarrow \dfrac{\Delta\lambda}{\lambda} = \dfrac{\Delta E}{E}$

$$\Delta\lambda = \frac{1240\text{ eV}}{1.8^2} \cdot 6.56\times10^{-3} = \underline{2.5\text{ nm}}.$$

Note: $\Delta\lambda$ is called the "natural line width" of the spectral line.
We can measure $\Delta\lambda \rightarrow$ lifetime Δt of excited state, as long as gas is low density, low temperature.
(Otherwise, thermal Doppler broadening of lines >> natural line width.
$\frac{1}{2}M_{atom}v^2 = 3/2\,kT$, $\dfrac{\Delta\lambda_{thermal}}{\lambda} = \sqrt{\dfrac{1\pm v/c}{1\mp v/c}} \approx \dfrac{2v}{c}$.)